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A family of generalized Fibonacci lattices: self-similarity and scaling of the wavefunction

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Abstract. A family of generalized Fibonacci lattices, which are locally isomorphic and exhibit a peculiar type of self-similarity, have been found to exist by shifting the position of the strip *(initial phase) in the projection method.* The necessary and sufficient conditions of generating such self-similar quasiperiodic lattices are established. The power-law growth behaviour of the wavefunction at $E = 0$ in the off-diagonal model defined on some of these lattices has been analysed. It is shown that, although various structures resulting from different initial phases are in a local isomorphism class, they lead to a variety of maximum exponents of power for the scaling of the wavefunction.

1. Introduction

The experimental discovery of the quasicrystal phase in metallic alloys [1], together with the realization of a quasiperiodic (QP) superlattice [2], has generated considerable interest in studying one-dimensional (1D) QP systems. In particular, much attention has been devoted to the systems with QP potentials based on the Fibonacci sequence, which provides a kind of prototype model for studying QP systems (see, e.g., [3] and references therein). Starting from the Fibonacci sequence, many generalizations have been proposed [4], mainly by generalizing the substitution rule that is characteristic for the QP Fibonacci sequence. The advantage of the generalization along this line lies in the fact that the resultant systems possess self-similarity so that one can exploit the renormalization-group (RG) technique introduced by Kohmoto, Kadanoff and Tang (KKT) [5] to work out many physical properties as well as the scaling of the electronic wavefunction and energy spectrum. However, not all the substitution rules generate sequences with both self-similarity and quasiperiodicity. The condition under which the substitution rule will generate a sequence that possesses quasiperiodicity has been discussed by many authors (see [3] and references therein). Another direction of generalizing the Fibonacci sequence, which has received less attention, is to keep the quasiperiodicity by using the standard projection method [6], or, equivalently, through the following algebraic technique:

$$f_k = \left\lfloor \frac{k+1}{1+\omega} + \theta_0 \right\rfloor - \left\lfloor \frac{k}{1+\omega} + \theta_0 \right\rfloor \tag{1}$$

where ω is a positive irrational number and is supposed to be less than 1 without loss of generality, k is an integer, and θ_0 is the initial phase, denoting the shift in position of the

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strip in the projection method [7]. Obviously, equation (1) generates binary QP sequences composed of 1s and 0s. When $\omega = \omega_1 = (\sqrt{5}-1)/2$, the inverse of the golden mean, and $\theta_0 = 0$, starting with $k = 1$, equation (1) produces the ordinary semi-infinite Fibonacci sequence. The sequences made by (1) are QP, but they are not all self-similar, i.e. some of them cannot be constructed with a certain substitution rule so that KKTs RG approach [5] seems to be inapplicable. So, an analogous question is that under what condition the sequence made by (1) will be self-similar. For the case with $\theta_0 = 0$ it has been shown [8] that when and only when ω is a quadratic irrational number† can a sequence made by (1) be self-similar.

Now a natural question is whether the self-similarity of the sequence is preserved when the initial phase $\theta_0 \neq 0$. In this paper, we consider the shifted precious mean (PM) sequences, namely the QP sequences made by (1) with non-vanishing θ_0 and ω being a PM number [9, 10]

$$\omega_n = \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{n + \ddots}}} \equiv [n, n, n, \ldots] \qquad n = 1, 2, 3, \ldots . \tag{2}$$

It is shown that the necessary and sufficient conditions for a shifted PM sequence to be self-similar is that θ_0 has the form

$$\theta_0 = \frac{1}{1+\omega_n} \frac{N + M\omega_n}{p} \tag{3}$$

where N, M and p are integers. The shifted PM sequences with θ_0 given by (3) exhibit a particular type of self-similarity, so that all of them form a natural family of generalized Fibonacci sequences.

The study of the initial phase θ_0 and its effect on the structural property of the QP sequence seems to be trivial, as the sequences with vanishing and non-vanishing θ_0 are locally isomorphic so that one may argue that there is no difference in physical characteristics, in spite of the different types of self-similarity. This, however, turns out to be incorrect. To illustrate some examples that different structures due to variant initial phases may have different physical features, we study the scaling behaviour of the wavefunction at $E = 0$ for an off-diagonal tight-binding model defined on some different sequences. It is shown that the different structures arising from various initial phases, although locally isomorphic, may cause the variety of the maximum exponents of power for the power-law growth of the wavefunction. The initial phase is therefore physically meaningful. With the same logic, one may expect that for more general 1D, two-dimensional (2D) and three-dimensional (3D) quasicrystals, a variety of structures appear as well, due to the different initial phases (the initial phase for higher-dimensional quasicrystals is the shift in position of the hyperprism in the projection method). In addition, some variant physical characteristics may result from these different structures, even when such different structures themselves are locally isomorphic among each other.

The rest of the paper is organized as follows. In section 2 the necessary and sufficient conditions for a shifted PM sequence to be self-similar are established. In section 3, we study the scaling behaviour for the amplitude of the wavefunction at $E = 0$ for an off-diagonal tight-binding model defined on some different sequences. The maximum exponent of power for the power-law growth of the wavefunction are calculated analytically. It is shown that various structures obtained from different initial phases may cause the variety

† A quadratic irrational number is a real number which is the solution to a quadratic algebraic equation with integer coefficients.

of the physical characteristics, although the structures themselves are locally isomorphic. Finally, concluding remarks are made in section 4.

2. Self-similarity

In this section, we shall show that the necessary and sufficient condition for a shifted PM sequence to be self-similar is that θ_0 has the form (3).

Before going on, some preliminaries are recalled first. For a binary QP sequence S made by (1) with arbitrary ω and θ_0, the following self-similarity transformation (deflation operation) $(D_n - T)$:

$$\begin{array}{l}(D_n - T)\ 1 = 1^n 0 \\ (D_n - T)\ 0 = 1\end{array} \qquad n = 1, 2, 3, \ldots \tag{4}$$

with 1^n standing for the concatenation of n 1s, corresponds to a transformation on the values of ω and θ_0 by

$$\omega^{(1)} = \frac{1}{n + \omega} \qquad \theta_0^{(1)} = -\omega^{(1)} \theta_0 . \tag{5}$$

That is to say, the sequence $S^{(1)} = (D_n - T)S$ can be made by (1) with $\omega^{(1)}$ and $\theta_0^{(1)}$. When $\omega = \omega_n$ and $\theta_0 = 0$, it follows from

$$\omega_n = \frac{1}{n + \omega_n} \tag{6}$$

that the sequence S generated by (1) is invariant under arbitrary times of deflation operations $(D_n - T)$. This is the case for the ordinary PM sequences [9, 10].

Now we turn to the shifted PM sequence, i.e. the case with $\omega = \omega_n$ but $\theta_0 \neq 0$. For such a case, in order that the sequence S be self-similar, one must expect that S is invariant under a finite l times self-similarity transformation $(D_n - T)$. Other types of transformation (substitution) are not appropriate, because they will change the value of ω (see equation (5)) and thus modify the ratio of the number of 1s to that of 0s, leading to a different sequence. As a result, if a shifted PM sequence S preserves self-similarity, one must have

$$S^{(l)} \equiv (D_n - T)^l\, S = S \tag{7}$$

where l is a finite integer. By noticing that

$$\omega_n^{(l')} = [\overbrace{n, \ldots, n}^{l'}, \omega_n] = \omega_n \qquad l' = 1, 2, 3, \ldots \tag{8}$$

it is not difficult to observe that a shifted PM sequence S is self-similar if and only if there exist a finite integer l such that the following relation is satisfied:

$$\theta_0^{(l)} \equiv (-\omega_n)^l\, \theta_0 = \theta_0 + \frac{K}{1 + \omega_n} \quad \pmod 1 \tag{9}$$

where the K is an integer, denoting a finite shift in the index k of the sequence. Taking into account the relation

$$(-\omega_n)^l = A_{l-1} - \omega_n A_l \tag{10}$$

where A_l's obey the following recursion relation:

$$A_{l+1} = n A_l + A_{l-1} \qquad (l \geqslant 1) \quad \text{with } A_0 = 0 \text{ and } A_1 = 1 \tag{11}$$

one easily comes to the conclusion that the necessary condition for a shifted PM sequence to be self-similar is that θ_0 can be expressed in the form (3). Thus, it is a rather trivial statement that the initial phase of a self-similar shifted PM sequence must be of the form (3). Taking the contraposition, one concludes that the shifted PM sequence with any other θ_0 does not have any self-similarity, in the sense that we cannot find any self-similarity transformation that makes the sequence invariant. In the following, we shall show that the converse is also true, i.e. the shifted PM sequence with θ_0 of the form (3) has self-similarity.

When θ_0 can be cast into form (3), equation (9) reads

$$(-\omega_n)^l \times \frac{1}{1+\omega_n}\frac{N+M\omega_n}{p} - \frac{1}{1+\omega_n}\frac{N+M\omega_n}{p} - i_x = \frac{K}{1+\omega_n} \tag{12}$$

where i_x is an integer. By using (10), the above equation reduces to

$$\begin{aligned}(A_{l+1}-1)M - A_l N &= i_x p \\ (A_{l-1}-1)N - A_l M &= (i_x + K)p\,.\end{aligned} \tag{13}$$

In other words, if one can find a finite l that satisfies (13), then the sequence S will be invariant under l times deflation operations, $(D_n - T)^l S = S$. On the other hand, by paying attention to (11), it is easy to observe that (13) is fulfilled whenever there exists a finite l such that $A_{l-1} - 1$ and A_l are both multiples of the integer p. Therefore, the problem of proving that the condition (3) is sufficient for a shifted PM sequence to be self-similar reduces to the problem of finding a finite l which makes both $A_{l-1} - 1$ and A_l multiples of p. In the following we present the proof for the existence of such an l.

Let us consider the case with positive p, while the case with negative p can be discussed in an analogous way. As $p = 1$ is a trivial case, in the following we assume $p > 1$. Define two sets of integers, $\{x_l\}$ and $\{y_l\}$, by

$$\begin{aligned} x_l &\equiv A_l \pmod p &\quad &\text{with} \quad 0 \leqslant x_l \leqslant p-1 \\ y_l &\equiv A_{l-1} - 1 \pmod p &\quad &\text{with} \quad -1 \leqslant y_l \leqslant p-2\,.\end{aligned} \tag{14}$$

It follows from the definitions that the existence of a finite l that makes $(x_l, y_l) = (0, 0)$ equivalent to the existence of a finite l such that both $A_{l-1} - 1$ and A_l are multiples of p, while the latter guarantees that the shifted PM sequence S is self-similar in the sense of $(D_n - T)^l S = S$.

With the use of (11), it is straightforward to derive the recursion relation for x_l and y_l:

$$x_{l+1} = nx_l + y_l + 1 \pmod p \qquad y_{l+1} = x_l - 1\,. \tag{15}$$

From the initial conditions $A_0 = 0$ and $A_1 = 1$, it follows that

$$x_1 = 1 \qquad y_1 = -1\,. \tag{16}$$

So in order to prove the existence of self-similarity, one has to show that starting with $(x_1,\ y_1) = (1, -1)$, one can get $(x_l,\ y_l) = (0, 0)$, for finite l, by the recursion transformation (15). To this end, let us first notice two characteristics of the transformation (15). The first one is that (15) is a one-to-one transformation, i.e. no two different pairs of (x, y), say $(x_{l-1}^{(1)}, y_{l-1}^{(1)})$ and $(x_{l-1}^{(2)}, y_{l-1}^{(2)})$, for example, can be transformed to the same (x_l, y_l) by (15). The second characteristic for the transformation (15) lies in the fact that starting with any initial conditions, as one carries out the transformation (15) further and further, either a finite cycle or a fixed point will be found in the (x, y) space, because the region for the allowed values of x_l and y_l is finite (see equation (14)). Note also that neither the cycle nor the fixed point can contain any dangling tail because of the one-to-one characteristic of transformation (15). Now let us start with $(x, y) = (0, 0)$. By a single time of transformation (15), we are led to the point $(x, y) = (1, -1)$ in the (x, y) space. Vice versa, if one starts with $(x, y) = (1, -1)$, then he will definitely get to $(0, 0)$ within a finite number of transformations (15), because of the above-mentioned two characteristics of the

Table 1. The S-sequences for some both-infinite shifted PM sequences associated with the initial phase θ_0 that can be expressed in the form (3). The quantity m is a positive integer. The bracket '( )' in the S-sequence denotes the unit of cycle, whereas the symbol '|' seperates two semi-infinite sequences extending to the right and to the left. The cycle length L_S for the S-sequence and the cycle length L_M for the corresponding transfer matrix (periodic M-series) are also listed. The maximum exponents of power β for the power-law scaling of the wavefunction at $E = 0$ are presented in the last column, with $\beta_0^{(n)}$ given by (41) and (51), for even and odd n, respectively.

n	θ_0	S-sequence	L_S	L_M	β
2	0^+	$\ldots)(S_3S_2)(S_2S_1)S_1^2S_0 \mid S_1(S_1S_0)(S_2S_1)(\ldots$	1	2	$\beta_0^{(2)}$
2	$-\frac{1}{2}$	$\ldots)(S_3)(S_2)(S_1)S_1 \mid (S_0)(S_1)(S_2)(\ldots$	1	2	$2\beta_0^{(2)}$
2	$-\frac{1}{3}$	$\ldots)(S_8S_7S_6S_6S_5S_5S_4S_3)S_1 \mid (S_1S_0S_2S_1S_2S_3S_4S_7)(\ldots$	8	8	$\frac{3}{2}\beta_0^{(2)}$
2	$\frac{1}{4}$	$\ldots)(S_2S_1S_1S_0) \mid S_1(S_3S_2S_4S_3)(\ldots$	4	4	$\beta_0^{(2)}$
3	0^+	$\ldots)(S_3^2S_2)(S_2^2S_1)S_1^3S_0 \mid S_1(S_1^2S_0)(S_2^2S_1)(\ldots$	1	6	$\beta_0^{(3)}$
3	$-\frac{1}{2}$	$\ldots)(S_4S_3S_2S_2^2S_1)S_1S_1 \mid (S_1S_0S_2^2S_1S_2)(\ldots$	3	6	$\frac{3}{2}\beta_0^{(3)}$
3	$-\frac{1}{3}$	$\ldots)(S_4S_4S_3)(S_2S_2S_1)S_1 \mid (S_1S_0S_1)(S_3S_2S_3)(\ldots$	2	6	$\beta_0^{(3)}$
$2m+2$	0^+	$\ldots)(S_3^{n-1}S_2)(S_2^{n-1}S_1)S_1^nS_0 \mid S_1(S_1^{n-1}S_0)(S_2^{n-1}S_1)(\ldots$	1	2	$\beta_0^{(2m+2)}$
$2m+2$	$-\frac{n-1}{n}$	$\ldots)(S_3^{n-2}S_2S_1)S_1^{n-1} \mid (S_0S_2^{n-2}S_1)(\ldots$	2	2	$2\beta_0^{(2m+2)}$
$2m+1$	0^+	$\ldots)(S_3^{n-1}S_2)(S_2^{n-1}S_1)S_1^nS_0 \mid S_1(S_1^{n-1}S_0)(S_2^{n-1}S_1)(\ldots$	1	6	$\beta_0^{(2m+1)}$
$4m+3$	$-\frac{1}{2(1+\omega_n)^2}$	$\ldots)(S_4^{2m}S_3S_3^{4m+1}S_2S_2^{4m+2}S_1)S_1^{2m+1} \mid (S_1^{2m+2}S_0)(S_4^{2m+2}S_3)(\ldots$	3	6	$2\beta_0^{(4m+3)}$
$4m+1$	$-\frac{1}{2}$	$\ldots)(S_4^{2m}S_3S_3^{2m-1}S_2S_2^{4m}S_1)S_1^{2m+1} \mid (S_1^{2m}S_0S_2^{4m}S_1S_3^{2m-1}S_2)(\ldots$	3	6	$2\beta_0^{(4m+1)}$

transformation. This, in fact, implies that there exists a finite l such that both $A_{l-1} - 1$ and A_l are multiples of p. From the above discussion, one easily sees that the same arguments are also valid for a negative p. This ends our proof for the sufficient condition.

Therefore, it is finally concluded that the necessary and sufficient condition for a shifted PM sequence to be self-similar is that θ_0 can be expressed in the form (3).

To see the specific self-similarity directly, we have decomposed the infinite-shifted PM sequence into a so-called S-sequence [7], which is composed of a finite-order PM sequence S_l defined by the inflation scheme [9, 10]

$$S_{l+1} = S_l^n S_{l-1} \qquad (l \geqslant 1) \quad \text{with} \quad S_0 = 0 \quad \text{and} \quad S_1 = 1 \tag{17}$$

where S_l^n denotes the concatenation of n S_l's. The decomposition rules are described in appendix. Table 1 shows the S-sequences for some infinite-shifted PM sequences associated with θ_0 of the form (3). In the table, the symbol '|' separates two semi-infinite sequences starting from $k = 1$ up to positive infinity and starting from $k = 0$ down to minus infinity (see equation (1)). The bracket '( )' in the S-sequence denotes the unit of cycle and helps to show the self-similarity of the sequence, i.e. the sequence is invariant under L_S times of deflation operations (4). For example, for the case with $n = 2$ and $\theta_0 = \frac{1}{4}$, the S-sequence will be

$$\ldots)(S_{10}S_9S_9S_8)(S_6S_5S_5S_4)(S_2S_1S_1S_0)|S_1(S_3S_2S_4S_3)(S_7S_6S_8S_7)(S_{11}S_{10}S_{12}S_{11})(\ldots. \tag{18}$$

While in the table, only the central part

$$\ldots)(S_2S_1S_1S_0)|S_1(S_3S_2S_4S_3)(\ldots \tag{19}$$

is shown, because the side parts are simply the $L_S = 4$ times deflation of the adjacent unit of cycle, e.g.

$$\begin{aligned} (S_6S_5S_5S_4) &= (D_2 - T)^4(S_2S_1S_1S_0) \\ (S_7S_6S_8S_7) &= (D_2 - T)^4(S_3S_2S_4S_3) \end{aligned} \tag{20}$$

where we have made use of $(D_n - T)^i S_l^j = S_{l+i}^j$. Notice that the same value of L_S appears naturally on both sides. In addition, one can easily derive

$$(D_2 - T)^4 S_1 = S_5 = (S_2 S_1 S_1 S_0) S_1 (S_3 S_2 S_4 S_3) . \tag{21}$$

The entire sequence is therefore self-similar in the sense that it coincides with itself by a finite L_S ordinary deflation operations (4). In fact, the quantity L_S is the least integer l which makes $(D_n - T)^l S = S$. In the rest of this paper, L_S will be called the cycle length for the S-sequence.

From table 1, it can be seen that the self-similarity is preserved whenever θ_0 can be cast into form (3), whereas different θ_0 may give rise to various types of self-similarity (see, the different values of L_S's in table 1).

3. Scaling of the wavefunction

In the last section, it has been shown that the shifted PM sequences with various initial phases may have different types of self-similarity (see, the different values of L_s's in table 1). These different structures are, in fact, locally isomorphic [11]. As a result, one may ask whether there is any difference in physical characteristics among these structures. Figure 1 shows the numerical results of the absolute square of the wavefunction $|\psi(k)|^2$ as a function of the site index k, at $E = 0$ in four finite off-diagonal model systems with vanishing and non-vanishing θ_0. It is clearly seen that the case with $\theta_0 \neq 0$ has a different feature from that of the case with $\theta_0 = 0$. In this section, the maximum exponents of power for the power-law scaling of the wavefunction at $E = 0$ of some model systems are calculated analytically, via a straightforward extension of the method of [10, 12], to confirm these different features.

The model is described by the off-diagonal tight-binding Hamiltonian

$$H = \sum_{k=0}^{\infty} t_{k+1} (|k\rangle\langle k+1| + |k+1\rangle\langle k|) \tag{22}$$

where $\{|k\rangle\}$ denote an orthonormalized set of bases characterized by the lattice sites $\{k\}$. The transfer energy t_k is taken to be t_a (t_b) if f_k is 1 (0) according to (1). The homogeneous equation

$$(H - E)\Psi(E) = 0 \tag{23}$$

for a given energy E can be reconstructed by using the transfer matrix T as follows

$$\begin{bmatrix} \psi(k+1) \\ \psi(k) \end{bmatrix} = M(k) \begin{bmatrix} \psi(1) \\ \psi(0) \end{bmatrix} \qquad M(k) = T(k+1,k)T(k,k-1)\ldots T(2,1) \tag{24}$$

where $\psi(k)$ denotes the value of the wavefunction $\Psi(E)$ at site k and the transfer matrix $T(k+1,k)$ is given by

$$T(k+1,k) \equiv T(t_{k+1}, t_k) = \begin{pmatrix} E/t_{k+1} & -t_k/t_{k+1} \\ 1 & 0 \end{pmatrix} \qquad (k = 1, 2, 3, \ldots) . \tag{25}$$

In order to take advantage of the deflation symmetry (self-similarity) of the lattice, it is natural to choose the following two basic transfer matrices:

$$B = T(t_a, t_a) \qquad A = T(t_a, t_b) T(t_b, t_a) B^{n-1} \tag{26}$$

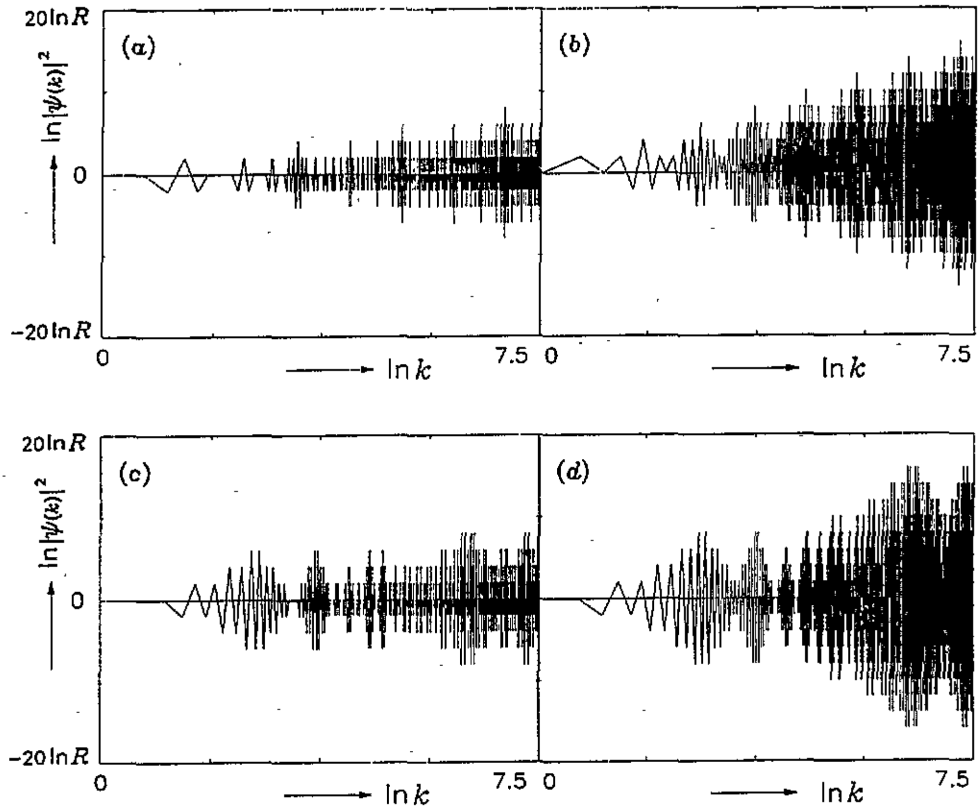

Figure 1. The ln–ln plot of the absolute square $|\psi(k)|^2$ of the wavefunction at $E = 0$ in the off-diagonal model (22), as a function of the site index k up to $k = 1800$, for the cases with (a) $n = 2$, $\theta_0 = 0$, (b) $n = 2$, $\theta_0 = -\frac{1}{2}$, (c) $n = 3$, $\theta_0 = 0$, and (d) the case with $n = 3$, $\theta_0 = -(\omega_n + 5)/18$. The transfer energies have been chosen as $t_b/t_a = R$, and the boundary condition $[\psi(0), \psi(1)] = (1, e^{i\phi})$.

so that the transfer matrix M_l for the ordinary finite PM lattice S_l of l generation can be obtained from the recurring relation

$$M_{l+1} = M_{l-1} M_l^n \tag{27}$$

together with the initial conditions $M_1 = B$ and $M_2 = A$. The two basic transfer matrices A and B connect basic blocks $a_l \equiv 1^n0$ and $b_l \equiv 1$ in the QP lattice. To be specific, through the product of the matrices A's and B's, one can only calculate the values of the wavefunction at the right end sites of the basic blocks a_1 and b_1, but not those at arbitrary lattice sites. However, as the sizes of blocks a_1 and b_1 are finite, the scaling behaviour of the wavefunction can be well determined by its values at the right end sites of such blocks. Therefore, in the rest of this section, we focus our attention on the analysis of the values of wavefunction at such right end sites. This enables us to deal solely with the product of the transfer matrices A's and B's.

At $E = 0$, the centre of the entire energy spectrum for the off-diagonal model, the two basic transfer matrices become

$$A = A^*(B^*)^{n-1} \qquad B = B^* \tag{28}$$

with

$$A^* = \begin{pmatrix} -R & 0 \\ 0 & -1/R \end{pmatrix} \qquad B^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{29}$$

and $R = t_b/t_a$. The simplicity of the form of the two basic matrices allows us to discuss the scaling behaviour of the wavefunction, analytically, for any shifted PM lattice that preserves the deflation symmetry.

Before discussing the scaling of the wavefunction, let us first explore some characteristics of the transfer matrix. As is shown in the appendix, the semi-infinite shifted PM sequence associated with any initial phase θ_0 can be decomposed into an S-sequence

$$S_{l_1+1}^{n_1} S_{l_1} S_{l_2+1}^{n_2} S_{l_2} S_{l_3+1}^{n_3} S_{l_3} \cdots \tag{30}$$

with $n_i \geqslant 0$ and $l_i \geqslant 0$ being integers, so the transfer matrix for the corresponding semi-infinite lattice can be written as

$$\ldots M_{l_3} M_{l_3+1}^{n_3} M_{l_2} M_{l_2+1}^{n_2} M_{l_1} M_{l_1+1}^{n_1} . \tag{31}$$

When the S-sequence has the cyclic structure with cycle length L_S (as was shown in table 1) and the transfer matrix M_l is of l_M-cycle, with $M_{l+l_M} = M_l$, the transfer matrix for the semi-infinite-shifted PM lattice will take the form of the following periodic M-series:

$$\ldots M_{l_i} M_{l_i+1}^{n_i}][M_{l_f} M_{l_f+1}^{n_f} \ldots M_{l_i} M_{l_i+1}^{n_i}][M_{l_f} M_{l_f+1}^{n_f} \ldots M_{l_i} M_{l_i+1}^{n_i}] \qquad (1 \leqslant l_i,\ l_f < \infty) \tag{32}$$

plus an additional term $M_{l_F} M_{l_F+1}^{n_F} \ldots M_{l_1} M_{l_1+1}^{n_1}$ at the right end. The additional term can be removed by a suitable *finite* shift of the initial point of the sequence and thus has no effect on the scaling of the wavefunction. The periodic M-series (32) coincides with itself by L_M times of the ordinary deflation operations, $(D_n - T)^{L_M}$, plus an insertion of the unit of cycle $[M_{l_f} M_{l_f+1}^{n_f} \ldots M_{l_i} M_{l_i+1}^{n_i}] \equiv X$ into the starting point (right end). The quantity L_M is called the cycle length for the M-series. It is the least common multiple of the cycle length l_M for the transfer matrix M_l and the cycle length L_S for the S-sequence.

Now we shall focus our discussion on some examples.

(i) n is even and $\theta_0 = 0$.

In this case, the semi-infinite PM sequence can be decomposed as

$$S_{\theta_0=0} = S_\infty = S_1(S_1^{n-1} S_0)(S_2^{n-1} S_1)(S_3^{n-1} S_2)(S_4^{n-1} \ldots \tag{33}$$

with $L_S = 1$. The two basic transfer matrices read

$$A = (-1)^{(n-2)/2} A^* B^* \qquad B = B^* .$$

The transfer matrix M_l is of two-cycle with $M_{l+2} = \pm M_l$ so that $L_M = 2$. After removing the finite initial part $S_1(S_1^d S_0)$ from (33), the transfer matrix for the remainder of the semi-infinite lattice,

$$(S_2^{n-1} S_1)(S_3^{n-1} S_2)(S_4^{n-1} S_3)(S_5^{n-1} S_4)(S_6^{n-1} \ldots \tag{34}$$

has the form of periodic M-series

$$\ldots][M_2 M_1^{n-1} M_1 M_2^{n-1}][M_2 M_1^{n-1} M_1 M_2^{n-1}] . \tag{35}$$

Here and in the rest of this section, we drop the sign of the transfer matrix for simplicity, since we are interested in the absolute value of the wavefunction. The value of the wavefunction at the right end site of the Jth basic block on lattice (34) can be written as

$$\begin{bmatrix} \psi(k_J+1) \\ \psi(k_J) \end{bmatrix} = M(k_J) \begin{bmatrix} \psi(1) \\ \psi(0) \end{bmatrix} = \hat{M}_J \begin{bmatrix} \psi(1) \\ \psi(0) \end{bmatrix} \tag{36}$$

where k_J labels the right end site of the Jth basic block, and $\hat{M}_J$ is the product of J basic transfer matrices A's and B's. By noticing that $A^2 = B^2 = -1$, it is possible to write $\hat{M}_J$ as

$$\hat{M}_J = g(q, r) = B^q (AB)^r \qquad q = 0, 1 \qquad r = \ldots, -2, -1, 0, 1, 2, \ldots \tag{37}$$

where we have once again neglected the change of sign for simplicity. If one considers open boundary conditions with $\psi(0) = 1$ and $\psi(1) = e^{j\phi}$, then the absolute value of wavefunction at the right end site of the Jth basic block on lattice (34) is given by

$$\begin{bmatrix} |\psi(k_J + 1)| \\ |\psi(k_J)| \end{bmatrix} = \begin{pmatrix} |R|^{\pm r} \\ |R|^{\mp r} \end{pmatrix} \tag{38}$$

with the sign before r dependent on the value of q.

Consider any two neighbouring basic blocks, J and $J+1$, on lattice (34). The matrices $\hat{M}_J$ and $\hat{M}_{J+1}$ determine the wavefunction at the right end sites of blocks J and $J+1$, respectively, where $\hat{M}_{J+1}$ is either $Bg(q, r)$ or $Ag(q, r)$. Making $L_M = 2$ times of ordinary deflation operations (4), we have

$$\begin{aligned} Bg(q, r) &\rightarrow BA^n g(q, r) X \\ Ag(q, r) &\rightarrow A(BA^n)^n g(q, r) X . \end{aligned} \tag{39}$$

The emergence of the matrix X, which is the transfer matrix for $(S_2^{n-1} S_1)(S_3^{n-1} S_2)$, is due to a shift of starting site by $(S_2^{n-1} S_1)(S_3^{n-1} S_2)$ between sequence (34) and the one after $L_M = 2$ deflation operations

$$(S_4^{n-1} S_3)(S_5^{n-1} S_4)(S_6^{n-1} S_5)(S_7^{n-1} S_6)(S_8^{n-1} \ldots . \tag{40}$$

After L_M deflation operations, the position of the block J changes to J', and $(J+1)$ to $(J+1)'$. The wavefunction at the right end site of block J' is determined by $\hat{M}_{J'} = g(q, r)X$, while that at the right end site of block $(J+1)'$ determined by either $\hat{M}_{(J+1)'} = BA^n g(q, r)X = Bg(q, r)X$ or $\hat{M}_{(J+1)'} = A(BA^n)^n g(q, r)X = Ag(q, r)X$, depending on block $(J+1)$ being either b_1 or a_1. Because blocks J' and $(J+1)'$ are now no longer neighbours, some new values of the wavefunction will appear between blocks J' and $(J+1)'$. Paying attention to (38) and using the analysis method presented in [10, 12], one can easily come to the conclusion that the wavefunction at $E = 0$ grows at most by a power law, $\psi(k) \sim k^\beta$, with the maximum exponent of power given by

$$\beta^{(n)}_{\theta_0=0} \equiv \beta^{(n)}_0 = -\frac{\ln |R|}{2 \ln \omega_n} \qquad \text{for even } n . \tag{41}$$

In deriving (41), use has been made of the fact that

$$X = M_2 M_3^{n-1} M_1 M_2^{n-1} = 1 \tag{42}$$

and L_M deflation operations, $(D_n - T)^{L_M}$, rescales the length of the lattice by $(1/\omega_n)^{L_M}$.

(ii) n is even and $\theta_0 = -(n-1)/n$.

In this case, the shifted PM sequence should be

$$S_{\theta_0=-(n-1)/n} = \left(S_0 S_2^{n-2} S_1\right)\left(S_2 S_4^{n-2} S_3\right)\left(S_4 S_6^{n-2} S_5\right)\left(S_6 \ldots \right. \tag{43}$$

with $L_S = 2$. After removing the initial part S_0, the transfer matrix for the remainder

$$\left(S_2^{n-2} S_1 S_2\right)\left(S_4^{n-2} S_3 S_4\right)\left(S_6^{n-2} S_5 S_6\right)\left(S_8^{n-2} \ldots \right. \tag{44}$$

reads

$$\ldots]\left[M_2 M_1 M_2^{n-2}\right]\left[M_2 M_1 M_2^{n-2}\right] . \tag{45}$$

With a similar discussion to case (i), it is easy to find that the important difference is that

$$X = M_2 M_1 M_2^{n-2} = AB \tag{46}$$

in place of (42). So the maximum exponent of power for the scaling of the wavefunction can be found to be

$$\beta^{(n)}_{\theta_0=-(n-1)/n} = -\frac{2\ln|R|}{2\ln\omega_n} = 2\beta_0^{(n)} \qquad \text{for even } n\,. \tag{47}$$

(iii) n is odd and $\theta_0 = 0$.
In this case, the semi-infinite PM sequence can be decomposed as (33). However, the two basic transfer matrices reduce to

$$A = (-1)^{(n-1)/2} A^* \qquad B = B^*\,.$$

The transfer matrix M_l is of six-cycle, $M_{l+6} = M_l$, so that $L_M = 6$. The transfer matrix for the semi-infinite lattice (34) is given by

$$\begin{aligned}\ldots]\big[M_6 M_7^{n-1} M_5 M_6^{n-1} M_4 M_5^{n-1} M_3 M_4^{n-1} M_2 M_3^{n-1} M_1 M_2^{n-1}\big]\\ \times\big[M_6 M_7^{n-1} M_5 M_6^{n-1} M_4 M_5^{n-1} M_3 M_4^{n-1} M_2 M_3^{n-1} M_1 M_2^{n-1}\big].\end{aligned} \tag{48}$$

The relations $(AB)^2 = B^2 = -1$ imply that $\hat{M}_J$ can be cast in the form

$$\hat{M}_J = g(q,r) = B^q A^r \qquad q = 0, 1 \qquad r = \ldots, -2, -1, 0, 1, 2, \ldots\,. \tag{49}$$

After making $L_M = 6$ ordinary deflation operations, $(D_n - T)^6$, we have a similar but more complicated relation as (39), with

$$X = M_6 M_7^{n-1} M_5 M_6^{n-1} M_4 M_5^{n-1} M_3 M_4^{n-1} M_2 M_3^{n-1} M_1 M_2^{n-1} = 1\,. \tag{50}$$

A discussion analogous to the former cases yields

$$\beta^{(n)}_{\theta_0=0} \equiv \beta_0^{(n)} = -\frac{(n+1)\ln|R|}{6\ln\omega_n} \qquad \text{for odd } n\,. \tag{51}$$

(iv) n is odd and $\theta_0 = -\frac{1}{2}$.
In this case, the semi-infinite shifted PM sequence is decomposed as

$$S_{\theta_0=-\frac{1}{2}} = \big(S_1^m S_0 S_2^{2m} S_1 S_3^{m-1} S_2\big)\big(S_4^m S_3 S_5^{2m} S_4 S_6^{m-1} S_5\big)\big(S_7^m S_6 \ldots \tag{52}$$

with $m = (n-1)/2 > 0$ and $L_S = 3$. The significant difference from case (iii) is that, in place of (50), X becomes

$$X = M_6 M_7^m M_5 M_6^{m-1} M_4 M_5^{2m} M_3 M_4^m M_2 M_3^{m-1} M_1 M_2^{2m} \tag{53}$$

which suggests

$$X = \begin{cases} A^{n+1} & \text{for even } m \\ A^{n-1} & \text{for odd } m\,. \end{cases} \tag{54}$$

It follows straightforwardly from a similar analysis that

$$\beta^{(n)}_{\theta_0=-\frac{1}{2}} = \begin{cases} -\dfrac{2(n+1)\ln|R|}{6\ln\omega_n} = 2\beta_0^{(n)} & \text{for even } m \\ -\dfrac{2n\ln|R|}{6\ln\omega_n} & \text{for odd } m\,. \end{cases} \tag{55}$$

Some other examples, for which variant maximum exponents of power for the power-law growth of the wavefunction are obtained due to different initial phases, are shown in table 1.

4. Conclusions

Binary QP sequences made by (1) with $\omega = \omega_n$ and non-vanishing initial phase θ_0, or, more visually, the binary QP sequences generated by the standard projection method [6] associated with $\tan\varphi = \omega_n$ and a shift in the position of the strip by θ_0 [7], have been shown to preserve self-similarity if and only if θ_0 can be cast into the form (3). Different types of self-similarity have been found to exist depending on the value of θ_0. Despite the difference in self-similarity, these QP sequences are locally isomorphic. As a consequence, one may believe that they will not produce any difference in physical characteristics. This, in fact, turns out to be incorrect, as has been found by studying the scaling behaviour of the wavefunction at $E = 0$ in an off-diagonal model. Our analytical results have shown that, depending on the value of θ_0, there exist various maximum exponents of power for the power-law scaling behaviour of the wavefunction. The variety of the power-law exponent for the wavefunction suggests that different types of system-size dependence of the resistance may exist.

The variety of the self-similar QP structures in a local isomorphism class and the variety of physical characteristics caused does not seem to be specific only to the generalized Fibonacci QP lattices. A similar feature can also be expected in more general 1D, 2D and 3D quasicrystals.

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Appendix.

In this appendix, we describe how to decompose an infinite shifted PM sequence into an S-sequence, consisting of ordinary PM sequences of finite order, S_l, defined by the inflation scheme (17).

Let us first concentrate on the right semi-infinite sequence starting from $k = 1$ up to positive infinity in (1). When $\theta_0 = 0$, the semi-infinite sequence is simply the limit of S_l as $l \to +\infty$ made by the inflation scheme (17). When θ_0 is small but finite, the new sequence will be identical to the ordinary PM sequence S_∞ only up to some finite k. If the initial part of the new sequence matches $S_{l+1}^{n'} S_l$ ($l \geqslant 0$, $n' = 0, 1, \ldots, n-1$) for its entire length, then it is said that the corresponding θ_0 generates $S_{l+1}^{n'} S_l$. It is not difficult to note that, whenever $S_{l+1}^{n'} S_l$ is generated, $S_{l+1}^{n''} S_l$ and $S_{l+1}^{n''}$, with $n'' \leqslant n'$, are also generated. As a result, for each value of θ_0 there must be a maximum value $l_{\rm m}$ of l and a maximum value $n_{\rm m}$ of n' for which $S_{l_{\rm m}+1}^{n_{\rm m}} S_{l_{\rm m}}$ is generated but $S_{l_{\rm m}+1}^{n_{\rm m}+\eta}$, with η a positive integer, is not generated. We shall say that $S_{l_{\rm m}+1}^{n_{\rm m}} S_{l_{\rm m}}$ is *guaranteed* by the initial phase. In other words, the guaranteed sequence is the longest generated sequence. For any θ_0, that is located inside the interval $[-1/\Delta, \omega_n/\Delta)$, with $\Delta = 1 + \omega_n$, the guaranteed finite sequence can be worked out as

$$\begin{aligned} &S_1^m S_0 \quad \text{for} \quad -\frac{\xi_m}{\Delta} \leqslant \theta_0 < -\frac{\xi_{m+1}}{\Delta} \qquad m = 0, 1, \ldots, n-1 \\ &S_2^m S_1 \quad \text{for} \quad -\frac{(-\omega_n)\xi_{m+1}}{\Delta} \leqslant \theta_0 < -\frac{(-\omega_n)\xi_m}{\Delta} \qquad m = 0, 1, \ldots, n-1 \end{aligned} \tag{A1}$$

and

$$S_{2j+1} \quad \text{for} \quad -\frac{(-\omega_n)^{2j}}{\Delta} \leqslant \theta_0 < -\frac{(-\omega_n)^{2j}\xi_1}{\Delta} < 0 \qquad j \geqslant 1$$
$$S_{2j+2} \quad \text{for} \quad 0 < -\frac{(-\omega_n)^{2j+1}\xi_1}{\Delta} \leqslant \theta_0 < -\frac{(-\omega_n)^{2j+1}}{\Delta} \qquad j \geqslant 1 \tag{A2}$$

$$S^m_{2j+2}S_{2j+1} \quad \text{for} \quad 0 < -\frac{(-\omega_n)^{2j+1}\xi_{m+1}}{\Delta} \leqslant \theta_0 < -\frac{(-\omega_n)^{2j+1}\xi_m}{\Delta}$$
$$m = 1, \ldots, n-1 \qquad j \geqslant 1 \tag{A3}$$
$$S^m_{2j+1}S_{2j} \quad \text{for} \quad -\frac{(-\omega_n)^{2j}\xi_m}{\Delta} \leqslant \theta_0 < -\frac{(-\omega_n)^{2j}\xi_{m+1}}{\Delta} < 0$$
$$m = 1, \ldots, n-1 \qquad j \geqslant 1 \tag{A4}$$

where $\xi_m = 1 - m\omega_n$. For the purpose of sequence decomposition, it is not always convenient to associate the longest generated sequence (the guaranteed sequence) $S^{n_m}_{l_m+1}S_{l_m}$ to θ_0. In fact, the idea of the sequence decomposition is that, after selecting a generated sequence, say $S^{n_1}_{l_1+1}S_{l_1}$, for example, as an element in the S-sequence for certain θ_0, one must treat $k = n_1 P_{l_1+1} + P_{l_1} + 1$ as the new starting point $k' = 1$, so that one can get a new initial phase and determine the next element in the S-sequence. Here P_l is the number of digits in the lth-order PM sequence S_l, defined by the following recursion relation:

$$P_{l+1} = nP_l + P_{l-1} \qquad (l \geqslant 1) \quad \text{with} \quad P_0 = P_1 = 1\,. \tag{A5}$$

To be more specific, after selecting $S^{n_1}_{l_1+1}S_{l_1}$ as the first element in the S-sequence, the *remainder of the shifted* PM *sequence should be regarded as a new shifted* PM *sequence with* the new initial phase θ'_0 given by

$$\theta'_0 \equiv \theta^{(1)}_0 = F\left(\theta_0 + \frac{n_1 P_{l_1+1} + P_{l_1}}{\Delta}\right) \tag{A6}$$

where

$$F(x) = \begin{cases} x - \lfloor x \rfloor & \text{for} \quad x - \lfloor x \rfloor < \omega_n/\Delta \\ x - \lfloor x \rfloor - 1 & \text{for} \quad x - \lfloor x \rfloor > \omega_n/\Delta \end{cases} \tag{A7}$$

with $\lfloor x \rfloor$ denoting the integer part of x. Note that by using the function $F(x)$, θ'_0 is once again located inside the interval $[-1/\Delta,\ \omega_n/\Delta)$. In order to ensure that the element of the type $S^{n'}_{l+1}S_l$ appears in increasing order of l in the S-sequence (which will be of central importance in discussing the self-similarity of the infinite sequence), the element in the S-sequence should be chosen carefully to make the new initial phase θ'_0 satisfy $|\theta'_0| < |\theta_0|$. For $n_\mathrm{m} \neq 0$, selecting the guaranteed sequence as an element in the S-sequence does leave us with a smaller $|\theta'_0|$. On the other hand, for $n_\mathrm{m} = 0$ and $l_\mathrm{m} \geqslant 3$, as in (A2), one must choose a shorter generated sequence $S_{l_\mathrm{m}-1}$, instead of selecting the guaranteed sequence S_{l_m}, as an element in the S-sequence in order to guarantee $|\theta'_0| < |\theta_0|$. With this rule in mind, the element in the S-sequence is determined according to the following rules:

$$S^m_{2j+2}S_{2j+1} \quad \text{for} \quad 0 < -\frac{(-\omega_n)^{2j+1}\xi_{m+1}}{\Delta} \leqslant \theta_0 < -\frac{(-\omega_n)^{2j+1}\xi_m}{\Delta}$$
$$m = 0, 1, \ldots, n-1 \qquad j \geqslant 0 \tag{A8}$$
$$S^m_{2j+1}S_{2j} \quad \text{for} \quad -\frac{(-\omega_n)^{2j}\xi_m}{\Delta} \leqslant \theta_0 < -\frac{(-\omega_n)^{2j}\xi_{m+1}}{\Delta} < 0$$
$$m = 0, 1, \ldots, n-1 \qquad j \geqslant 0 \tag{A9}$$

and the new initial phase can be worked out as

$$\theta_0' = \theta_0 + \frac{(-\omega_n)^{2j}\xi_m}{\Delta} \quad \text{for} \quad -\frac{(-\omega_n)^{2j}\xi_m}{\Delta} \leqslant \theta_0 < -\frac{(-\omega_n)^{2j}\xi_{m+1}}{\Delta} < 0$$
$$\theta_0' = \theta_0 + \frac{(-\omega_n)^{2j+1}\xi_m}{\Delta} \quad \text{for} \quad 0 < -\frac{(-\omega_n)^{2j+1}\xi_{m+1}}{\Delta} \leqslant \theta_0 < -\frac{(-\omega_n)^{2j+1}\xi_m}{\Delta}$$
$$m = 0, 1, \ldots, n-1 \qquad j \geqslant 0. \tag{A10}$$

It is easy to see that

$$0 \leqslant \theta_0' < -\frac{(-\omega_n)^{2j+1}}{\Delta} \quad \text{for} \quad -\frac{(-\omega_n)^{2j}\xi_m}{\Delta} \leqslant \theta_0 < -\frac{(-\omega_n)^{2j}\xi_{m+1}}{\Delta} < 0$$
$$-\frac{(-\omega_n)^{2j+2}}{\Delta} \leqslant \theta_0' < 0 \quad \text{for} \quad 0 < -\frac{(-\omega_n)^{2j+1}\xi_{m+1}}{\Delta} \leqslant \theta_0 < -\frac{(-\omega_n)^{2j+1}\xi_m}{\Delta}$$
$$m = 0, 1, \ldots, n-1 \qquad j \geqslant 0 \tag{A11}$$

so the new initial phase θ_0' is less than $|\theta_0|$ in absolute value and will guarantee an element of the type $S_{l+1}^{n'} S_l$ in the S-sequence with increasing order of l from the left to the right.

We are now ready to show how to decompose a shifted PM sequence with any θ_0 located inside the interval $(-1/\Delta \leqslant \theta_0 < \omega_n/\Delta)$ into an S-sequence. If $\theta_0 = 0$, then the complete PM sequence follows, so that the decomposition ends up with the sole member S_∞ in the S-sequence. If $\theta_0 \neq 0$, on the other hand, we may register $S_{l_1+1}^{n_1} S_{l_1}$ as the first element in the S-sequence provided that θ_0 lies between $-(-\omega_n)^{l_1}\xi_{n_1}/\Delta$ and $-(-\omega_n)^{l_1}\xi_{n_1+1}/\Delta$. At the same time, we treat $k = n_1 P_{l_1+1} + P_{l_1} + 1$ as the new starting point $k' = 1$ and the new initial phase θ_0' is determined by (A10). If $\theta_0' = 0$, then S_∞ is registered as the second and the last element in the S-sequence, and the decomposition is completed. If $\theta_0' \neq 0$, on the other hand, we have to determine the second element from the value of θ_0' by (A8) and (A9). Then the new initial phase θ_0'' is calculated by (A10). From θ_0'', we can obtain the third element in the S-sequence. This process is repeated infinitely until we find a vanishing initial phase. At each step, the element of the type $S_{l+1}^{n'} S_l$ is determined by (A8) and (A9), and the new initial phase $\theta_0^{(m+1)}$ is found from the previous one $\theta_0^{(m)}$ through (A10). The procedure uniquely determines the decomposition of any shifted PM sequence into an S-sequence. At each step of the decomposition, the relation between the new initial phase $\theta_0^{(m+1)}$ and the previous one $\theta_0^{(m)}$ guarantees that the elements of the type $S_{l+1}^{n'} S_l$ appear in increasing order of l from the left to the right, with the increment of l between any two adjacent elements being always odd. Furthermore, from (A11) we know that S_∞ never follows $S_{l+1}^{n'} S_l$ of an odd index l. In fact, if these properties are to be required of the S-sequence, the decomposition will be unique.

As for the left semi-infinite sequence starting from $k = 0$ down to minus infinity made by (1), with some decomposition rules similar to (A9) and (A10), one can also decompose it into a succession of finite PM sequences. However, for the sequence extending to the left, we choose to ensure that the element of the type $S_{l+1}^{n'} S_l$ appears in the S-sequence in increasing order of l from the *right* to the *left*, rather than from the left to the right. This makes it convenient for us to observe the self-similarity of the shifted PM sequence with θ_0 of the form (3), as was discussed in section 2 and shown in table 1.

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